# Low-temperature phase transition in the three-state Potts glass

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The low-temperature instability of one-step replica symmetry breaking (1RSB) phase in three-state Potts spin glass is obtained explicitly. The temperature of the instability is higher than the temperature where the 1RSB entropy becomes negative. The conjecture of the possibility of the low-temperature full RSB is supported.

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During last decades the so called 1RSB (one-step replica-symmetry breaking) models: the $p$-state Potts spin glasses and $p$-spin spin glasses as well as their soft-spin versions are in the focus of investigations in spin glass theory. It is usually believed that the 1RSB solution if stable in the vicinity of its appearance remains stable till zero temperature. In this paper we demonstrate explicitly the low-temperature instability of 1RSB solution for the three-state Potts spin glass using its representation in terms of quadrupole operators.

The $p$-state Potts spin-glass model is a lattice model where each lattice site carries a Potts spin $\sigma_i$ which can take one of the $p$ values $\sigma_i=0,1,\ldots,p-1$ and the interaction Hamiltonian is

$$H=-\frac{p}{2}\sum_{i\neq j} J_{ij}\delta_{\sigma_i\sigma_j}, \tag{1}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol. Thus, a pair $\{\sigma_i,\sigma_j\}$ contributes an energy $-J_{ij}$ if $\sigma_i=\sigma_j$ and zero otherwise. The interactions $J_{ij}$ are quenched random variables described by a Gaussian distribution

$$P(J_{ij})=(\sqrt{2\pi}J)^{-1}\exp[-(J_{ij}-J_0)^2/2J^2].$$

The Potts glass with an infinite-range interaction $J_0=\tilde{J}_0/N$, $J=\tilde{J}/N^{1/2}$ has been studied in Refs. [1–10]. The short-range version has been considered in Refs. [11–13] and is a subject of intense investigation through computer simulations [14]. The soft-spin version of Potts glass has also been suggested as a starting point for a theory of structural glasses and the transition from the metastable fluid to the glass state [15]. The Potts glass may also serve as a model for orientational glasses in molecular crystals and cluster glasses where a strong single-site anisotropy restricts the orientation of the appropriate molecular group to $p$ distinct directions.

The three-state Potts spin glass (PG) is somehow intermediate system between Sherrington-Kirkpatrick (SK) glass ($p=2$) and "canonical" 1RSB glasses ($p\geqslant 4$). In three-state PG there is no reflection symmetry and 1RSB solution was shown [5,8] to be stable in the vicinity of the RS transition temperature (which coincides with that of 1RSB transition) against the higher stages of RSB, but the static transition is continuous, which is not general properties of 1RSB models.

The stability of the 1RSB solution till zero temperature was established rigorously by Crisanti and Sommers [16] in the case of the spherical $p$-spin interaction spin-glass model. The spin-glass phase of the spherical $p$-spin model is described exactly by a step order parameter function, i.e., the 1RSB is the most general solution within the Parisi RSB scheme.

As far as we know there is only one paper where the phase transition from 1RSB to full RSB (FRSB) phase was established. Using perturbations around known solutions for the cases of $p=2+\epsilon$ and $p\to\infty$ glass E. Gardner [17] showed for Ising $p$-spin glass that 1RSB solution is unstable at *very* low temperature. The second transition leads to a phase described by a continuous order parameter of FRSB function $q(x)$.

The RSB solution for the mean field three-state PG was considered in Refs. [5,7,10]. It was shown [5,7] that 1RSB solution is stable in the vicinity of phase transition. In Ref. [5] it was supposed that at lower temperature another spin-glass phase appears which differs from the 1RSB in the nature of the correlations among the many degenerate ground states of the system. However, this second phase transition was not found in the thorough investigation by De Santis, Parisi, and Ritort [10]. There are no observations whatsoever that would indicate that the short-range system also has two successive phase transitions [12–14]. It is worth to note that this possible second transition (often called Gardner transition) to low-temperature FRSB phase was usually regarded as an inessencial, and somehow exotic phenomenon. However, in Ref. [18] it was shown that the metastable states which are relevant for the out-of-equilibrium dynamics of such systems are always in a FRSB phase. This renewes the interest to the low-temperature behavior of the systems without reflection symmetry.

Let us consider now the system of particles on lattice sites $i,j$ with the Hamiltonian [6,7]

$$H=-\frac{1}{2}\sum_{i\neq j} J_{ij}(Q_iQ_j+V_iV_j), \tag{2}$$

where $Q=3S_z^2-2$, $V=\sqrt{3}(S_x^2-S_y^2)$, $S=1, S_z=1,0,-1$; $Q^2=2-Q$, $V^2=2+Q, QV=VQ=V$. A particle quadrupole moment is the second-rank tensorial operator with five com-

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ponents. In the principal axes frame only two of them remain: $Q$ and $V$. In the subspace $\mathbf{S}=1$ the following equality holds:

$$\frac{1}{6}(Q_mQ_n+V_mV_n+2)=\delta_{mn}.$$

This equality shows the equivalence of Eq. (2) to the $p=3$ Potts Hamiltonian (1). We shall assume that $J_{ij}$ are distributed following the Gaussian law with zero mean,

$$P(J_{ij})=(\sqrt{2\pi}J)^{-1}\exp[-J_{ij}^2/2J^2],$$

and $J=\tilde{J}/N^{1/2}$.

Using the standard procedure of the replica method, we get the expression for the free energy, corresponding to the Hamiltonian (2) [6]:

$$\frac{\langle F\rangle_J}{NkT}=-\lim_{n\to 0}\frac{1}{n}\max\Bigg\{-2t^2-t^2\sum_{(\alpha\beta)}(q^{\alpha\beta})^2-\frac{t^2}{2}\sum_{\alpha}(x^{\alpha})^2 + \ln Tr\exp\Bigg[t^2\sum_{(\alpha\beta)}q^{\alpha\beta}(Q^{\alpha}Q^{\beta}+V^{\alpha}V^{\beta}) + t^2\sum_{\alpha}Q^{\alpha}x^{\alpha}\Bigg]\Bigg\}. \tag{3}$$

Here $(\alpha\beta)$ means the sum over the couples of replicas, $n$ is the number of replicas, $t=\tilde{J}/kT$, $x^{\alpha}=\langle\langle Q^{\alpha}\rangle\rangle$, $q^{\alpha\beta}=\frac{1}{2}\langle\langle Q^{\alpha}Q^{\beta}+V^{\alpha}V^{\beta}\rangle\rangle$ are the order parameters, $\langle\langle\cdots\rangle\rangle$ means thermodynamic average and average over disorder.

The RS solution of saddle-point equations gives $x=0$, $q\neq 0$ for $T<T_c$ where $kT_c=2J$ [2,6]. This solution is unstable against 1RSB at $T_c$. Following Parisi scheme we carry out the first step of RSB by dividing the $n$ replicas into $n/m$ groups of $m$ replica and setting $q_{\alpha\beta}=q_1$ if $\alpha$ and $\beta$ belong to the same group and $q_{\alpha\beta}=q_0$ otherwise. In the limit $n\to 0$ the parameter $m$ is constrained to the range $0\leqslant m\leqslant 1$. In the absence of an external field only two variables remain $q-q_0=v$ and $m$ and the free energy takes the form

$$\frac{\langle F\rangle_J}{NkT}=-2t^2+\frac{t^2}{2}(m-1)v^2+2t^2v - \frac{1}{m}\ln\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dz^G dy^G\Psi^m(\theta_1,\theta_2), \tag{4}$$

where

$$da^G=\frac{1}{\sqrt{2\pi}}da e^{-a^2/2},$$

$$\Psi=e^{-2\theta_1}+e^{\theta_1}(e^{\theta_2}+e^{-\theta_2}),$$

$$\theta_1=tz\sqrt{v},\qquad \theta_2=ty\sqrt{3v},$$

$v,m$ satisfy the saddle-point equations (in fact, the maximum conditions) for Eq. (4),

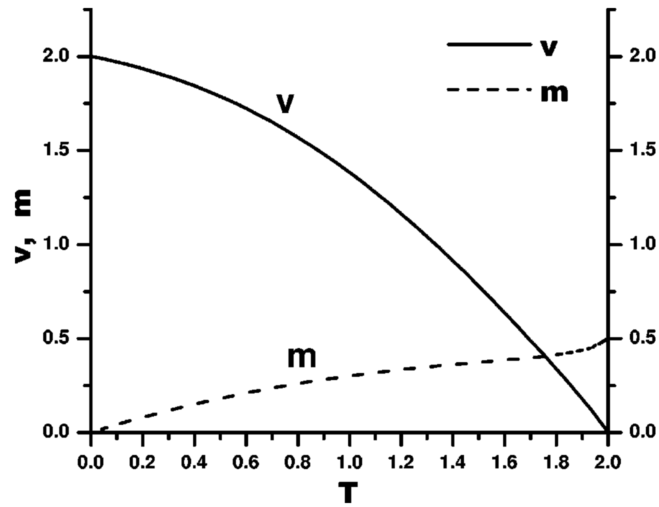


FIG. 1. Order parameters as functions of $T=kT/J$. (One must notice that $kT_c=2J$.)

$$v=\frac{1}{2}\frac{\langle\Psi^{m-2}((\Psi_1')^2+3(\Psi_2')^2)\rangle}{\langle\Psi^m\rangle}, \tag{5}$$

$$-\frac{t^2v^2m^2}{2}=\ln\langle\Psi^m\rangle-m\frac{\langle\Psi^m\ln\Psi\rangle}{\langle\Psi^m\rangle}. \tag{6}$$

Here

$$\Psi_i'=\frac{\partial\Psi}{\partial\theta_i}$$

and

$$\langle\cdots\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dz^G dy^G\cdots.$$

It is worth to note that our free energy (4) coincides with that given by Eq. (16) of Ref. [10] if we put there $p=3$, $q=v/2$, $\beta=2t$, introduce new variables $y_1=z_1, y_2=z_2-z_1, y_3=z_3-z_1$. This enables one to integrate explicitly over $y_1$ so that only two integrals remain. This simpler form increases the precision of calculations significantly.

The solution of Eqs. (5) and (6) close to $T_c$ in the form of step-function $v(m)$ was obtained in Refs. [5,7] and was shown to be stable near $T_c$ against further RSB. Now performing numerical maximization of Eq. (4) we obtain the values of $v$ and $m$ for all temperatures. The results are presented in Fig. 1 and do coincide with that of the paper Ref. [10] (as well as the results for dynamical transition temperature defined from the marginal stability condition). The corresponding free energy (4) changes the sign of the slope at low temperature, so that the entropy of the 1RSB solution becomes negative (see Fig. 2). To obtain the point where further RSB is necessary one has to consider the stability of the obtained 1RSB solution for $v$ and $m$ with respect to further replica breaking. Following Almeida and Thouless [20], we expand the free energy (3) around the 1RSB free energy (4). Stability requires that all eigenvalues of the stability ma-

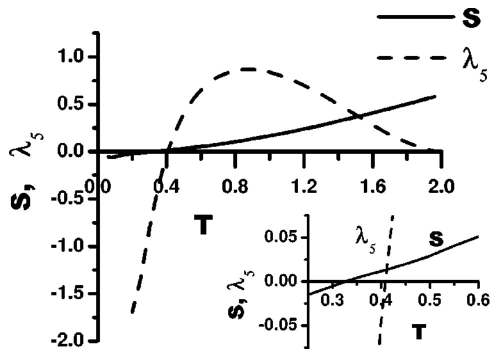


FIG. 2. $\lambda_{repl}$ and the entropy as functions of temperature ($T = kT/J$).

trix associated with fluctuations evaluated within 1RSB solution should be positive. In the limit $n \to 0$ we get six eigenvalues [7]:

$$\lambda_1 = P + 2(m-2)Q + \frac{1}{2}(m-2)(m-3)R, \tag{7}$$

$$\lambda_2 = P' + 2(m-1)Q' + (m-1)^2 R', \tag{8}$$

$$\lambda_3 = P + (m-4)Q - (m-3)R, \tag{9}$$

$$\lambda_4 = P' - 2Q' + R', \tag{10}$$

$$\lambda_5 = P - 2Q + R, \tag{11}$$

$$\lambda_6 = P' + (m-2)Q' + (1-m)R', \tag{12}$$

where

$$P = 4 - 2t^2[8 + 4v - 4v^2],$$

$$Q = -2t^2[4v - 4v^2 + t_3],$$

$$R = -2t^2[-4v^2 + r_4],$$

$$P' = 4 - 16t^2, \quad Q' = -8t^2 v, \quad R' = -4t^2 v^2,$$

$$t_3 = -\frac{\langle \Psi^{(m-3)}(\Psi_1')^3 \rangle}{\langle \Psi^m \rangle} + 9\frac{\langle \Psi^{(m-3)}\Psi_1'(\Psi_2')^2 \rangle}{\langle \Psi^m \rangle},$$

$$r_4 = \frac{\langle \Psi^{(m-4)}((\Psi_1')^2 + 3(\Psi_2')^2)^2 \rangle}{\langle \Psi^m \rangle}.$$

Five of these eigenvalues occur to be always positive. But $\lambda_5$—the replicon mode—changes sign at the temperature higher than that of the changing of the slope of the 1RSB free energy,

$$\lambda_{repl}(=\lambda_5) = 4 - 2t^2[8 - 4v + r_4 - 2t_3]. \tag{13}$$

In Fig. 2 the behavior of the $\lambda_{repl}$ along with the entropy as functions of temperature are presented. The instability point is $t_2 = 0.4$ and in this point the critical values of parameters are $v_c = 1.84$ and $m_c = 0.15$.

At the point $t_2$ it is necessary to perform further replica symmetry breaking and a transition to a different type of glass phase occurs. Further steps of RSB within Parisi scheme will result in a new transition between the 1RSB regime and a more complex regime. These two phases differ in the nature of the correlations among the many degenerate ground states of the system. We suppose that this complex regime at lower temperature can in fact be properly described by the FRSB Ansatz by Parisi as it was proposed in Ref. [5]. However, we do not exclude the possibility of obtaining the FRSB regime not immediately at the 1RSB instability point but through a sequence of higher order separate RSB phase transitions.

As some kind of indirect indication that zero temperature phase of our model (2) corresponds to FRSB the results of the paper Ref. [8] can be considered. The well-known result by Tanaka and Edwards [19] for the number of metastable states at zero temperature in SK spin glass is generalized there. In Ref. [19] the macroscopically large number $\langle N_s \rangle_{SK}$ of the metastable states at $T = 0$ was obtained

$$\langle N_s \rangle_{SK} = \exp[-N\Omega_{SK}],$$

where $\Omega_{SK} = -0.199\,23$. This result was obtained without any reference to RSB scheme and even replica approach. Using the method of Ref. [19] in the paper Ref. [8] the number of metastable states $\langle N_s \rangle_{3P}$ at $T = 0$ for the model (2) was obtained,

$$\langle N_s \rangle_{3P} = \exp[-N\Omega_{3P}],$$

with $\Omega_{3P} = \Omega_{SK} + \ln(\frac{3}{2})$, so that the "relative" number of metastable states (the part of all possible $p^N$ states) is the same as in SK model,

$$\frac{\exp[-N\Omega_{3P}]}{3^N} = \frac{\exp[-N\Omega_{SK}]}{2^N}. \tag{14}$$

It seems that this fact shows a similarity of the structure of zero temperature landscapes in these two models and supports the importance to look for FRSB phase in three-state PG.

To conclude, in this paper, using the quadrupole representation Eq. (2) for the three-state PG, we obtained explicitly the point of the instability of 1RSB solution. So, we give some additional support to the Gross, Kanter, Sompolinsky conjecture about the low-temperature behavior of the three-state Potts spin glass. We think that our success is based on the fact that we have fewer problems with precision at low-temperatures, because Eqs. (5) and (6) are simpler than the corresponding equations of Ref. [10].

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